MHD EQUATIONS FOR PARAMAGNETIC MEDIA

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The conservation laws are used to obtain phenomenologically the complete system of equations of motion of a conductive paramagnetic fluid in a magnetic field. In addition to the usual MHD equations (with additional terms accounting for the magnetization of the medium), this system includes the equation for the rate of change of the magnetic moment.

The hydrodynamic equations for a fluid with internal rotation have been obtained in [1] and extended in [2] to the case of the paramagnetic properties resulting from this rotation: here the fluid was considered nonconducting. The analysis of [2] is extended to the case of a fluid with nonzero electrical conductivity. This will be the same extension of MHD as the theory of [1, 2] is for conventional hydrodynamics.

§1. The intensity of the motion of the medium in question is characterized at each point by the hydrodynamic velocity v and by the bulk density K of the internal moment of momentum. The latter is connected with the bulk density $\mathbf{M}$ of the magnetic moment by the relation

$$\mathbf{M} = \gamma \mathbf{K}, \qquad (1.1)$$

where γ is the gyromagnetic ratio. The quantities K and $\mathbf{M}$ are the macroscopic characteristics of the electronic and nuclear motions, both orbital and spin.

We shall make use of the phenomenological scheme for the derivation of the equations of motion of a continuum, proposed by Landau and used in [3] and [1, 2]. We start from the following equations.

The macroscopic equations expressing the conservation of mass, energy, momentum (equation of motion), and moment of momentum are as follows:

$$\partial \rho / \partial t + \operatorname{div}(\rho \mathbf{v}) = 0, \qquad (1.2)$$

$$\partial E / \partial t + \operatorname{div} \mathbf{Q} = 0, \qquad (1.3)$$

$$\rho\left(\frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k}\right) = \frac{\partial \sigma_{ik}}{\partial x_k} - \frac{\partial p}{\partial x_i}, \qquad (1.4)$$

$$\frac{\partial}{\partial t}(L_{ik} + K_{ik}) + \frac{\partial G_{ikl}}{\partial x_l} = 0,$$

$$(L_{ik} = e_{ikl}L_l, \ K_{ik} = e_{ikl}K_l, \ \mathbf{L} = \rho(\mathbf{r} \times \mathbf{v})). \qquad (1.5)$$

Here E is the energy bulk density, Q and G_{ikl} are the energy flux density and moment of momentum density, σ_{ik} is the stress sensor, and p is the pressure.

The equations for the change of the internal moment of momentum and entropy are as follows:

$$\frac{\partial K_{ik}}{\partial t} + \frac{\partial}{\partial x_l}(v_l K_{ik}) = f_{ik}, \qquad (1.6)$$

$$\rho T(\partial s / \partial t + \mathbf{v} \nabla s) = F. \qquad (1.7)$$

From the field equations in a moving conductive medium [4]

$$\partial \mathbf{B} / \partial t = \operatorname{rot}(\mathbf{v} \times \mathbf{B}) - \lambda \operatorname{rot} \operatorname{rot} \mathbf{H},$$

$$(\lambda = c^2 / 4\pi\sigma), \qquad (1.8)$$

$$\operatorname{div} \mathbf{B} = 0 \qquad (\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}). \qquad (1.9)$$

Assuming that $L_{ik} = \rho(x_i v_k - x_k v_i)$, after simple calculations [1] we find from (1.4)–(1.6) that

$$f_{ik} = \sigma_{ki} - \sigma_{ik} - \partial g_{ikl} / \partial x_l$$

$$g_{ikl} = G_{ikl} - v_l(L_{ik} +$$

$$K_{ik}) + x_i \sigma_{kl} - x_k \sigma_{il} - p(x_i \delta_{kl} - x_k \delta_{il}) \qquad (1.10)$$

Here s is the entropy per unit mass, T is the absolute temperature, f_{ik} and F are the nonconvective parts of the variations of the corresponding quantifes (F is a dissipative function), B is the magnetic induction, and σ is the electrical conductivity of the fluid.

From (1.1), (1.6), and (1.10) follows the equation for the magnetization:

$$\frac{\partial M_{ik}}{\partial t} + \frac{\partial}{\partial x_l}(v_l M_{ik}) = -\gamma\left(\sigma_{ik} - \sigma_{ki} + \frac{\partial g_{ikl}}{\partial x_l}\right). \qquad (1.11)$$

It remains for us to find on what quantities and how E, Q, σ_{ik}, g_{ikl}, and F depend.

§2. The energy E is made up of the kinetic energy of the moving fluid, the field energy in the medium, and the internal energy of the paramagnetic substance:

$$E = \frac{\rho v^2}{2} + \frac{BH}{8\pi} + E_0. \qquad (2.1)$$

The known thermodynamic relation [4]

$$\frac{\partial E}{\partial \mathbf{B}} = \frac{\mathbf{H}}{4\pi} = \frac{\mathbf{B}}{4\pi} - \mathbf{M}.$$

permits writing E in the form

$$E = \frac{\rho v^2}{2} + \frac{B^2}{8\pi} - BM + U(\rho, s, M^2). \qquad (2.2)$$

The magnetization of a paramagnetic substance is always small. Therefore, in the expansion of U in powers of M^2 it is sufficient to retain terms of zero and first order:

$$U = U_0(\rho, s) + \frac{M^2}{2\varkappa}$$

Comparing this expression for E with (2.1), we find that

$$E_0 = U_0(\rho, s) + \frac{\mathbf{M}}{2\varkappa}(\mathbf{M} - \varkappa\mathbf{B}). \qquad (2.3)$$

The equilibrium condition E = min yields the equilibrium expression for the magnetization

$$\mathbf{M} = \varkappa\mathbf{B}, \qquad (2.4)$$

where $\varkappa > 0$ (paramagnetism). From the thermodynamic identity for the internal energy

$$dE_0 = \rho T ds + w d\rho +$$

$$+ \varkappa^{-1}(\mathbf{M} - \varkappa\mathbf{B})d\mathbf{M} - \mathbf{M}d\mathbf{B} + {}^1/{}_2 d(\mathbf{BM}), \qquad (2.5)$$

and the definition of enthalpy, $w = \rho^{-1}(E_0 + p)$, follows the expression for the differential pressure:

$$dp = -\rho T ds + \rho dw -$$
$$- \varkappa^{-1}(\mathbf{M} - \varkappa\mathbf{B})d\mathbf{M} + \mathbf{M}d\mathbf{B} - \tfrac{1}{2}d(\mathbf{BM}). \quad (2.6)$$

§3. Now let us find the expressions for the quantities Q, σ_{ik}, g_{ikl}, and F. We differentiate (2.1) with respect to time, making use of (2.5). In the resulting expression we substitute the time derivatives of ρ, v, s, B, and M, taken from (1.2), (1.4), (1.7), (1.8), and (1.11), and with the aid of (1.9) and (2.6) we obtain, after transformations (see [1, 2]).

$$\frac{\partial E}{\partial t} + \operatorname{div}\left\{ \rho\mathbf{v}\left(\frac{v^2}{2} + w\right) - (\sigma\mathbf{v}) + \right.$$
$$+ \mathbf{v}\frac{\mathbf{BH}}{8\pi} - \frac{\lambda}{4\pi}\mathbf{H}\times\operatorname{rot}\mathbf{H} +$$
$$\left. + \frac{\gamma}{\varkappa}(g, \mathbf{M} - \varkappa\mathbf{B}) \right\} =$$
$$= F - \frac{\lambda}{4\pi}(\operatorname{rot}\mathbf{H})^2 + \frac{\gamma}{\varkappa} g_{ik}\frac{\partial}{\partial x_h}(M_i - \varkappa B_i) -$$
$$- \left\{ \sigma_{ik} + \frac{\mathbf{M}}{\varkappa}(\mathbf{M} - \varkappa\mathbf{B})\delta_{ik} - \frac{1}{4\pi}\left(H_i B_k - \frac{\mathbf{BH}}{2}\delta_{ik}\right)\right\}\times$$
$$\times \left[\frac{\partial v_i}{\partial x_h} + \frac{\gamma}{\varkappa}(M_{ik} - \varkappa B_{ik})\right], \quad (3.1)$$

where $(\sigma v) \equiv \sigma_{ik}\nu_k$, $(g, \mathbf{M} - \varkappa\mathbf{B}) \equiv g_{ik}(M_k - \varkappa B_k)$.

Comparing (3.1) with (1.3), we conclude that Q is determined by the expression following the div symbol in the left-hand side of (3.1), and

$$F = \frac{\lambda}{4\pi}(\operatorname{rot}\mathbf{H})^2 - \frac{\gamma}{\varkappa} g_{ik}\frac{\partial}{\partial x_h}(M_i - \varkappa B_i) +$$
$$+ \left\{ \sigma_{ik} + \frac{\mathbf{M}}{\varkappa}(\mathbf{M} - \varkappa\mathbf{B})\delta_{ik} - \right.$$
$$\left. - \frac{1}{4\pi}\left(H_i B_k - \frac{\mathbf{BH}}{2}\delta_{ik}\right)\right]\left[\frac{\partial v_i}{\partial x_h} + \frac{\gamma}{\varkappa}(M_{ik} - \varkappa B_{ik})\right]. \quad (3.2)$$

For the further calculations it is convenient to write the stress tensor in the form of the sum of the symmetric and antisymmetric parts

$$\sigma_{ik} = S_{ik} + e_{ikl}A_l \quad (3.3)$$

and symmetrize all the terms in the right-hand side of (3.2). The condition $F > 0$ (law of increasing entropy) makes it possible to determine the form of A, S_{ik}, g_{ik}, and F.

Retaining in F only the terms which are quadratic in the quantities characterizing the deviation from equilibrium, we have

$$-2\gamma A =$$
$$= \gamma(\mathbf{M}\times\mathbf{B}) - \frac{1}{\tau}(\mathbf{M} - \varkappa\mathbf{B}) - \frac{\alpha\varkappa}{M^2}\mathbf{M}\times(\mathbf{M}\times\mathbf{B}), \quad (3.4)$$

$$S_{ik} = -\frac{\mathbf{M}}{\varkappa}(\mathbf{M} - \varkappa\mathbf{B})\delta_{ik} + \frac{1}{4\pi}\left(B_i B_k - \frac{\mathbf{BH}}{2}\delta_{ik}\right) -$$
$$- \frac{1}{2}(M_i B_k + M_k B_i) + \pi_{ik}, \quad (3.5)$$

$$-\gamma g_{ik} =$$
$$= \delta_1\left[\frac{\partial(M_i - \varkappa B_i)}{\partial x_h} + \frac{\partial(M_k - \varkappa B_k)}{\partial x_i} - \frac{2}{3}\frac{\partial M_l}{\partial x_l}\delta_{ik}\right] +$$

$$+ \delta_2\frac{\partial M_l}{\partial x_l}\delta_{ik} - \delta_3\left[\frac{\partial(M_k - \varkappa B_k)}{\partial x_i} - \frac{\partial(M_i - \varkappa B_i)}{\partial x_h}\right]$$

$$\pi_{ik} = \eta\left(\frac{\partial v_i}{\partial x_h} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\frac{\partial v_l}{\partial x_l}\delta_{ik}\right) + \zeta\frac{\partial v_l}{\partial x_l}\delta_{ik}. \quad (3.6)$$

Here π_{ik} is the usual viscous stress tensor, and α, τ, λ, δ_1, δ_2, $\delta_3 \geq 0$.

§4. Substituting the expressions for σ_{ik} and $g_{ikl} = e_{ikm}g_{ml}$ into (1.4) and (1.11), we obtain

$$\rho\left[\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v}\right] =$$
$$- \nabla\left[p + \frac{\mathbf{M}}{\varkappa}(\mathbf{M} - \varkappa\mathbf{B}) + \frac{\mathbf{BH}}{8\pi}\right] + \frac{1}{4\pi}(\mathbf{B}\nabla)\mathbf{H} +$$
$$+ \eta\nabla^2\mathbf{v} + (\zeta + \tfrac{1}{3}\eta)\nabla\operatorname{div}\mathbf{v} + \frac{1}{2\gamma\tau}\left[1 + \alpha\tau\varkappa\frac{(\mathbf{MB})}{M^2}\right]\times$$
$$\times \operatorname{rot}(\mathbf{M} - \varkappa\mathbf{B}) - \frac{\alpha\varkappa}{2\gamma}\mathbf{M}\times\nabla\left[\frac{(\mathbf{MB})}{M^2}\right],$$
$$\frac{\partial\mathbf{M}}{\partial t} + (\mathbf{v}\nabla)\mathbf{M} = \gamma(\mathbf{M}\times\mathbf{B}) - \mathbf{M}\operatorname{div}\mathbf{v} -$$
$$\frac{1}{\tau}(\mathbf{M} - \varkappa\mathbf{B}) - \frac{\alpha\varkappa}{M^2}\mathbf{M}\times(\mathbf{M}\times\mathbf{B}) -$$
$$- D_1\operatorname{rot}\operatorname{rot}(\mathbf{M} - \varkappa\mathbf{B}) + D_2\nabla\operatorname{div}\mathbf{M}$$
$$(D_1 \equiv \delta_1 + \delta_3, \quad D_2 \equiv \delta_2 + \tfrac{4}{3}\delta_1). \quad (4.1)$$

Equations (4.1), (1.2), (1.8), and (1.9) form the complete system of equations of motion of the medium in question; for $\varkappa = \mathbf{M} = 0$ they become the conventional MHD equations (4).

The following basic physical effects develop with interaction of the magnetic and hydrodynamic phenomena.

1°. Currents are induced in a conducting fluid moving in an applied magnetic field:

(a) the field of these currents alters the original magnetic field;

(b) interaction of the currents and the field creates an electromagnetic force which alters the original motion.

These two effects are the basis for the various phenomena which are studied in conventional MHD, where the magnetization of the fluid is not considered. Additional effects are associated with taking this magnetization into account.

2°. The applied field magnetizes the medium:

(c) interaction of the magnetic moment with the field leads to the appearance of additional ponderomotive forces in the equation of motion;

(d) the magnetic field of the currents induced by the motion of the fluid (see (a) above) alters the magnetization;

(e) the variation of the density of the medium resulting from compressibility is accompanied by a variation of the magnetic moment density.

One remark must be made in conclusion. In deriving the equations of motion, the use of (2.1) for the energy density E was essential. The hydrodynamic velocity v appears in (2.1) through $1/2\rho v^2$. But for a fluid with internal rotation v can also appear in E through the expression characterizing the relation between the internal and "external" rotations (with the angular velocity $\Omega = 1/2 \operatorname{rot} v$). The necessity for the relation follows from conservation of the total moment of momentum. The corresponding expression was found in [2] and has the form $K\Omega = M(\Omega/\gamma)$.

In the present study this expression has not been taken into account, since it was found in [2] that this would only lead to the appearance in

certain terms of (4.1) of the "gyromagnetic field" Ω/γ, summed with B, and estimates show that this term is small in comparison with B for all reasonable values of B and Ω.

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