EPJ E
Soft Matter and Biological Physics

EPJ.org
your physics journal

Eur. Phys. J. E (2015) **38**: 55 DOI 10.1140/epje/i2015-15055-x

Collision of viscoelastic bodies: Rigorous derivation of dissipative force

Denis S. Goldobin, Eugeniy A. Susloparov, Anastasiya V. Pimenova and Nikolai V. Brilliantov

edp sciences
Società Italiana di Fisica
Springer

Eur. Phys. J. E (2015) **38**: 55
DOI 10.1140/epje/i2015-15055-x

THE EUROPEAN PHYSICAL JOURNAL E

Regular Article

Collision of viscoelastic bodies: Rigorous derivation of dissipative force

Denis S. Goldobin[1,2,3], Eugeniy A. Susloparov[2], Anastasiya V. Pimenova[1], and Nikolai V. Brilliantov[3,a]

[1] Institute of Continuous Media Mechanics, UB RAS, Perm 614013, Russia
[2] Department of Theoretical Physics, Perm State University, Perm 614990, Russia
[3] Department of Mathematics, University of Leicester, Leicester LE1 7RH, UK

Received 29 December 2014 and Received in final form 20 April 2015
Published online: 22 June 2015 –

Abstract. We report a new theory of dissipative forces acting between colliding viscoelastic bodies. The impact velocity is assumed not to be large to neglect plastic deformations in the material and propagation of sound waves. We consider the general case of bodies of an arbitrary convex shape and of different materials. We develop a mathematically rigorous perturbation scheme to solve the continuum mechanics equations that deal with both displacement and displacement rate fields and accounts for the dissipation in the bulk of the material. The perturbative solution of these equations allows to go beyond the previously used quasi-static approximation and obtain the dissipative force. The derived force does not suffer from the inconsistencies of the quasi-static approximation, like the violation of the third Newton's law for the case of different materials, and depends on particle deformation and deformation rate.

1 Introduction

Granular materials are abundant in nature; they range from sand and powders on Earth to planetary rings and dust clouds in outer space [1–5]. These material exhibit very unusual properties, demonstrating solid-like, liquid-like or gas-like [6–9] behavior, depending on the external load or magnitude of agitation [10–12]. The physical reason for many unusual phenomena in granular media is the nature of inter-particles interactions there. Contrary to molecular or atomic systems, where particles interact only trough conservative, elastic forces, the interaction between granular particles include dissipative forces. This happens because the grains are themselves macroscopic bodies, which contain macroscopically large number of microscopic degrees of freedom. Hence, during an impact of such bodies their mechanical energy, associated with the translational or rotational motion, or with the elastic deformation of the particles' material, is partly transformed into the internal degrees of freedom, that is, into heat. In many applications however, the temperature increase of the grains is insignificant and may be neglected [6]. Obviously, for an adequate description of granular media one needs a quantitative model of inter-particles forces, which includes both elastic and dissipative components.

The elastic part of the inter-particle force is known for more than a century from the famous work of Hetrz [13]. He derived a mathematically rigorous result for the force acting between elastic bodies at a contact, provided the deformation of the bodies is small as compared to their size. In spite of a large importance for applications, a rigorous derivation of the dissipative force is still lacking. The existing phenomenological expressions for the dissipative force use either linear, *e.g.* [14,15] or quadratic [16] dependence of the force on the deformation rate. Neither of these dependencies is consistent with the experimental data, *e.g.* [14,17]. A derivation of the dissipative force from the first-principles has been undertaken in ref. [18]. A very restrictive approximation used in this work —the assumption that only shear deformations are important, substantially limits its applicability. A first "complete" derivation of the dissipative force between viscoelastic bodies has been done only recently [19]. It was based on the continuum mechanics equations and exploited a *quasi-static* approximation. It is assumed in this approximation that the displacement field in the bulk of colliding bodies coincides with that of a static contact [19]. The correct functional dependence of the dissipative force, derived in ref. [19] has been already suggested (without any rigorous mathematic derivation) in the earlier work of Kuwabara and Kono [20]. In the later studies [21, 22] a flaw in the derivation of the dissipative force of ref. [19] has been corrected. Still the restrictive assumption of the quasi-static approximation was used [21, 22].

[a] e-mail: nb144@le.ac.uk

Physically, the quasi-static approximation assumes the immediate response of the particle's material to the external load. Two conditions are to be fulfilled in order to make this approximation valid: i) the characteristic deformation rate should be much smaller than the speed of sound in the system and ii) microscopic relaxation time of the particle's material should be much shorter than the duration of the impact. The microscopic relaxation time quantifies the time needed for the material of a deformed body to respond to the applied load; in what follows we will give the detailed definition of this quantity. To go beyond the quasi-static approximation, that is, to take into account the deviation of the displacement field in the bulk of a deformed body from the static displacement field, we develop a perturbation approach based on a small parameter —the ratio of microscopic relaxation time and collision duration. In the most of applications this ratio is indeed small. Hence, we rigorously derive for the first time a dissipative force acting between viscoelastic particles. Although the quasi-static approximation is based on the physically plausible approach, it possesses some inconsistency, which is not so obvious for a collision of particles of the same material. At the same time when particles of different materials suffer an impact, the latter approximation predicts non-equal dissipative forces between the bodies; this violates the third Newton's law. Another inconsistency is related to the dependence of the dissipative force on the Poisson ratio —within the quasi-static approximation one obtains vanishing dissipative force for the materials with the Poisson ratio close to 1/2, which corresponds to much larger bulk modulus as compared to the shear one; this is definitely not physical. These difficulties of the quasi-static approximation are discussed in detail below.

Our new theory, based on the perturbation scheme, is mathematically rigorous and the obtained dissipative force is free from the above inconsistencies. In the present work we analyze a general case of an impact of viscoelastic bodies of an arbitrary shape and of different materials. The results for a more simple case of a collision of a sphere with un-deformable plane, which allows less involved derivation technique, has been reported earlier [23].

The rest of the paper is organized as follows. In the next sect. 2 we introduce the equation of motion of viscoelastic medium which we solve for the case of interest in the next sections. In sect. 3 the solution for the static contact is discussed; here we illustrate the general approach and derive the classical Hertz law. In sect. 4 the dynamic contact is addressed. We elaborate the perturbation scheme and using this scheme derive in sect. 5 the next-order solution. In sect. 6 we present our new theory for the dissipative force between colliding viscoelastic bodies. Finally, in sect. 7, we summarize our findings.

2 Equation of motion for viscoelastic medium

When two viscoelastic bodies are brought in a contact, so that they are deformed, an interaction force between the bodies arises. Generally, it contains the elastic and viscous parts; for a static contact, however, only the elastic force appears. To compute the forces, one needs to find a stress that emerges in the bodies and integrate the stress over the contact zone. The distribution of stress in the material is governed by the equation for a continuum medium which reads, see *e.g.* [24],

$$\rho\ddot{\mathbf{u}} = \boldsymbol{\nabla}\cdot\hat{\sigma} = \boldsymbol{\nabla}\cdot\left(\hat{\sigma}^{\mathrm{el}} + \hat{\sigma}^{\mathrm{v}}\right). \tag{1}$$

Here ρ is the material density, $\mathbf{u} = \mathbf{u}(\mathbf{r})$ is the displacement field in a point $\mathbf{r}$ and $\hat{\sigma}$ is the stress tensor, comprised of the elastic $\hat{\sigma}^{\mathrm{el}}$ and viscous part $\hat{\sigma}^{\mathrm{v}}$. The elastic stress linearly depends on the strain tensor,

$$u_{ij} = \frac{1}{2}\left(\nabla_i u_j + \nabla_j u_i\right),$$

and has the following form [24]:

$$\sigma_{ij}^{\mathrm{el}}(\mathbf{u}) = 2E_1\left(u_{ij} - \frac{1}{3}\delta_{ij}u_{ll}\right) + E_2\delta_{ij}u_{ll}. \tag{2}$$

Similarly, the viscous stress linearly depends on the strain *rate* tensor [24]:

$$\sigma_{ij}^{\mathrm{v}}(\dot{\mathbf{u}}) = 2\eta_1\left(\dot{u}_{ij} - \frac{1}{3}\delta_{ij}\dot{u}_{ll}\right) + \eta_2\delta_{ij}\dot{u}_{ll}. \tag{3}$$

Here $E_1 = \frac{Y}{2(1+\nu)}$ and $E_2 = \frac{Y}{3(1-2\nu)}$, with Y and ν being, respectively the Young modulus and Poisson ratio of the body material. η_1 and η_2 are the viscosity coefficients for the shear and bulk viscosity and i, j, l denote Cartesian coordinates; the Einstein's summation rule is applied.

The elastic deformation implies that, after separation of the contacting particles, they completely recover their initial form so that no plastic deformation remains. Only such deformations will be addressed below.

3 Static contact: Hertz theory

To introduce the notations and illustrate the derivation method we start with the static case, that is, we consider a time-independent contact of two convex bodies. We assume that only normal forces with respect to the contact area act between the particles. We place the coordinate system in the center of the contact region, where $x = y = z = 0$ (fig. 1). Let the displacement field in the upper body, located at $z > 0$, be $\mathbf{u}(\mathbf{r})$, while in the lower body, located at $z < 0$ be $\mathbf{w}(\mathbf{r})$. Then the deformation ξ which is equal to the sum of the compressions of the both bodies in the center of the contact zone is related to the z-components of the displacements of the upper and lower bodies' surfaces at the contact plane $u_z(x, y, 0)$ and $w_z(x, y, 0)$, see fig. 1. It may be shown [24] that for the bodies of arbitrary shape the following relation holds true:

$$B_1x^2 + B_2y^2 + u_z(x, y, 0) + w_z(x, y, 0) = \xi, \tag{4}$$

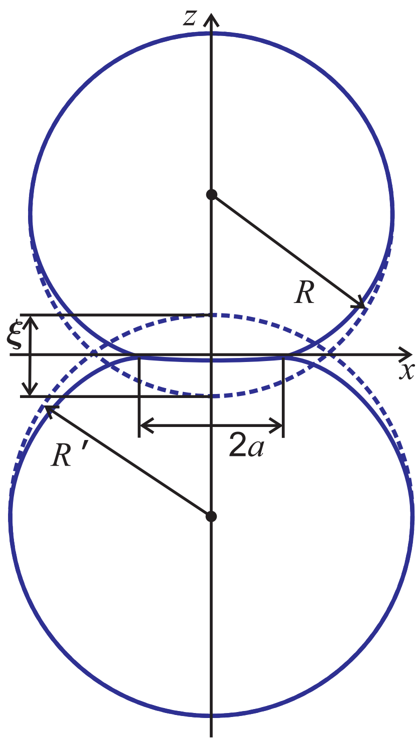

Fig. 1. Illustrates a simple case of a collision of two visco-elastic spheres in the according coordinate frame (the dashed profiles show undeformed bodies). Note that in the text a general case of arbitrary convex bodies is addressed.

where the constants B_1 and B_2 are related to the radii of curvature of the bodies' surfaces near the contact [24],

$$2\,(B_1+B_2)=\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_1'}+\frac{1}{R_2'}\,, \tag{5}$$

$$4\,(B_1-B_2)^2=\left(\frac{1}{R_1}-\frac{1}{R_2}\right)^2+\left(\frac{1}{R_1'}-\frac{1}{R_2'}\right)^2 +2\cos 2\varphi\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\left(\frac{1}{R_1'}-\frac{1}{R_2'}\right). \tag{6}$$

Here R_1, R_2 and R_1', R_2' are respectively the principal radii of curvature of the first and the second body at the point of contact and φ is the angle between the planes corresponding to the curvature radii R_1 and R_1'. Equations (4)–(6) describe the general case of the contact between two smooth bodies (see [24] for details). The physical meaning of (4) is easy to see for the case of a contact of a soft sphere of a radius R ($R_1=R_2=R$) with a hard, undeformed plane ($R_1'=R_2'=\infty$). In this case $B_1=B_2=1/2R$, the compressions of the sphere and of the plane are respectively $u_z(0,0,0)=\xi$ and $w_z=0$, and the surface of the sphere before the deformation is given by $z(x,y)=\frac{1}{2R}(x^2+y^2)$ for small z. Then the relation (4) may be recast in the flattened area into the form, $u_z(x,y)=\xi-z(x,y)$, which is the condition for a point $z(x,y)$ on the body's surface to touch the plane $z=0$.

While eq. (4) defines the displacement on the contact surface, the displacement fields in the bulk of the first (upper) and second (lower) bodies are determined by the following equations:

$$\boldsymbol{\nabla}\cdot\hat{\sigma}^{\rm el}(\mathbf{u})=0,\qquad \boldsymbol{\nabla}\cdot\hat{\sigma}^{\rm el}(\mathbf{w})=0. \tag{7}$$

Both equations may be solved by the same approach, therefore in what follows we consider the solution for the upper body with $z>0$. Using eq. (2) which relates the stress and strain tensors, we write

$$\nabla_j\sigma_{ij}^{\rm el}=E_1\Delta u_i+\left(E_2+\frac{1}{3}E_1\right)\nabla_i\nabla_j u_j=0, \tag{8}$$

where the elastic constants refer to the upper body (for the notation simplicity we do not add now the additional index specifying the body —it will be done later).

To solve the above equation we use the approach of [24] and write the solution as

$$\mathbf{u}=f^{(0)}\mathbf{e}_z+\nabla\varphi^{(0)}. \tag{9}$$

Here $\varphi^{(0)}=K^{(0)}zf^{(0)}+\psi^{(0)}$, where $K^{(0)}$ is some constant to be found and $f^{(0)}$ and $\psi^{(0)}$ are unknown harmonic functions. We assume the lack of tangential stress at the interface, which is *e.g.* fulfilled when the bodies at a contact are of the same material. Taking into account that

$$\Delta\mathbf{u}=\Delta\nabla\varphi^{(0)}=2K^{(0)}\nabla\frac{\partial f^{(0)}}{\partial z} \tag{10}$$

and

$$\nabla\cdot\mathbf{u}=(1+2K^{(0)})\frac{\partial f^{(0)}}{\partial z}\,, \tag{11}$$

we recast eq. (8) into the following form:

$$\nabla_j\sigma_{ij}^{\rm el}=\left[2E_1K^{(0)}+\left(1+2K^{(0)}\right)\left(E_2+\frac{E_1}{3}\right)\right]\nabla_i\frac{\partial f^{(0)}}{\partial z}=0, \tag{12}$$

which implies that

$$K^{(0)}=-\frac{1}{2}\frac{3E_2+E_1}{3E_2+4E_1}\,. \tag{13}$$

Consider now the boundary condition for the stress tensor. Obviously, on the free boundary all components of the stress vanish. In the contact region, located at the surface, $z=0$, the tangential components of the stress tensor $\sigma_{zx}^{\rm el}$ and $\sigma_{zy}^{\rm el}$ vanish as well, while the normal component of the stress tensor reads

$$\mathbf{n}\cdot\hat{\sigma}^{\rm el}=-\sigma_{zz}^{\rm el}=P_z, \tag{14}$$

where $\mathbf{n}=(0,0,-1)$ is the external normal to the upper body on the contact plane and P_z is the normal pressure acting on the contact surface. Therefore the boundary conditions have the following form:

$$\sigma_{zx}^{\rm el}\big|_{z=0}=0;\qquad \sigma_{zy}^{\rm el}\big|_{z=0}=0;\qquad \sigma_{zz}^{\rm el}\big|_{z=0}=-P_z. \tag{15}$$

Using the expression (2) for the elastic part of the stress tensor, together with the displacement vector (9) we recast the boundary conditions (15) into the form:

$$\frac{\partial}{\partial x}\left(\frac{3E_1}{4E_1+3E_2}f^{(0)}+2\frac{\partial\psi}{\partial z}\right)\bigg|_{z=0}=0, \tag{16}$$

$$\frac{\partial}{\partial y}\left(\frac{3E_1}{4E_1+3E_2}f^{(0)}+2\frac{\partial\psi}{\partial z}\right)\bigg|_{z=0}=0, \tag{17}$$

$$\frac{\partial}{\partial z}\left(\frac{3E_1}{4E_1+3E_2}f^{(0)}+2\frac{\partial\psi}{\partial z}\right)\bigg|_{z=0}=-\frac{P_z}{E_1}. \tag{18}$$

From equations (16) and (17) follows the relation between $f^{(0)}$ and $\frac{\partial\psi}{\partial z}$ at $z=0$:

$$\left(\frac{\partial\psi}{\partial z}+\frac{3}{2}\frac{E_1}{4E_1+3E_2}f^{(0)}\right)\bigg|_{z=0}=\text{const}=0. \tag{19}$$

The constant in the above relation equals to zero, since it holds true independently on the coordinate that is, also at the infinity; at the infinity, however, the deformation and thus the above functions vanish. Since $f^{(0)}$, ψ as well as $\partial\psi/\partial z$ are the harmonic functions, the condition that their linear combination vanishes on the boundary, eq. (19), implies that it is zero in the total domain, that is,

$$\frac{\partial\psi}{\partial z}=-\frac{3}{2}\frac{E_1}{4E_1+3E_2}f^{(0)}. \tag{20}$$

Substituting the last relation into (18) yields

$$\frac{\partial f^{(0)}}{\partial z}\bigg|_{z=0}=-\frac{4E_1+3E_2}{E_1(E_1+3E_2)}P_z. \tag{21}$$

Since $f^{(0)}$ is a harmonic function, one can use the relation between the normal derivative of a harmonic function on the surface and its value in the bulk, as it follows from the theory of harmonic functions (see *e.g.* [24,25]), hence we find

$$f^{(0)}(\mathbf{r})=\frac{4E_1+3E_2}{2\pi E_1(E_1+3E_2)}\iint_S\frac{P_z(x',y')\,\mathrm{d}x'\,\mathrm{d}y'}{|\mathbf{r}-\mathbf{r}'|}, \tag{22}$$

where S is the contact area. Using eq. (9) we can write z-component of the zero-order displacement at $z=0$ as

$$u_z|_{z=0}=(1+K^{(0)})f^{(0)}\Big|_{z=0}+\frac{\partial\psi}{\partial z}\bigg|_{z=0},$$

which together with (20) and definition of $K^{(0)}$, eq. (13) yields,

$$u_z|_{z=0}=\frac{1}{2}\,f^{(0)}\Big|_{z=0}. \tag{23}$$

If we now express E_1 and E_2 in terms of ν_1 and Y_1, where ν_1 and Y_1 are the according constants for the upper body (recall that we consider the upper contacting body) one obtains from eqs. (23), (22) and (15):

$$u_z(x,y,z=0)=-\frac{(1-\nu_1^2)}{\pi Y_1}\iint_S\frac{\sigma^{\mathrm{el}}_{zz}(x',y',z\!=\!0)\,\mathrm{d}x'\,\mathrm{d}y'}{|\mathbf{r}-\mathbf{r}'|}. \tag{24}$$

The same considerations may be performed for the lower body. Taking into account that the external normals for the upper and lower bodies as well as the exerted pressures are equal up to a minus sign ($\mathbf{n}=\mathbf{n}_{\mathrm{up}}=-\mathbf{n}_{\mathrm{low}}$, $P_z=P_{z,\mathrm{up}}=-P_{z,\mathrm{low}}$), we obtain,

$$w_z(x,y,z=0)=-\frac{(1-\nu_2^2)}{\pi Y_2}\iint_S\frac{\sigma^{\mathrm{el}}_{zz}(x',y',z\!=\!0)\,\mathrm{d}x'\,\mathrm{d}y'}{|\mathbf{r}-\mathbf{r}'|}. \tag{25}$$

Hence, with eqs. (24) and (25) the relation (4) takes the form

$$\frac{1}{\pi}\left(\frac{1-\nu_1^2}{Y_1}+\frac{1-\nu_2^2}{Y_2}\right)\iint_S\frac{P_z(x',y')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}x'\,\mathrm{d}y'=$$
$$\xi-B_1x^2-B_2y^2, \tag{26}$$

The last equation is an integral equation for the unknown function $P_z(x,y)$. We compare this equation with the mathematical identity [24],

$$\iint_S\frac{\mathrm{d}x'\,\mathrm{d}y'}{|\mathbf{r}-\mathbf{r}'|}\sqrt{1-\frac{x'^2}{a^2}-\frac{y'^2}{b^2}}=$$
$$\frac{\pi ab}{2}\int_0^\infty\left[1-\frac{x^2}{a^2+t}-\frac{y^2}{b^2+t}\right]\frac{\mathrm{d}t}{\sqrt{(a^2+t)(b^2+t)t}}, \tag{27}$$

where the integration in the left-hand side (l.h.s.) is performed over the elliptical area $x'^2/a^2+y'^2/b^2\le 1$. The l.h.s. of both eqs. (26) and (27), contain integrals of the same type, while the r.h.s. contain quadratic forms of the same type. Therefore, the contact area is an ellipse with the semi-axes a and b and the pressure is of the form

$$P_z(x,y)=\text{const}\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}.$$

The constant here may be found from the total elastic force F_{el} acting between the bodies. Integrating $P_z(x,y)$ over the contact area we get F_{el}, which then yields the constant. Hence we obtain

$$P_z(x,y)=\frac{3F_{\mathrm{el}}}{2\pi ab}\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}. \tag{28}$$

We substitute (28) into (26) and replace the double integration over the contact area by integration over the variable t, according to the above identity. Thus, we obtain an equation containing terms proportional to x^2, y^2 and a constant. Equating the corresponding coefficients we obtain

$$\xi=\frac{F_{\mathrm{el}}D}{\pi}\int_0^\infty\frac{\mathrm{d}t}{\sqrt{(a^2+t)(b^2+t)t}}=\frac{F_{\mathrm{el}}D}{\pi}\frac{N(\zeta)}{b}, \tag{29}$$

$$B_1=\frac{F_{\mathrm{el}}D}{\pi}\int_0^\infty\frac{\mathrm{d}t}{(a^2+t)\sqrt{(a^2+t)(b^2+t)t}}=\frac{F_{\mathrm{el}}D}{\pi}\frac{M(\zeta)}{a^2b}, \tag{30}$$

$$B_2=\frac{F_{\mathrm{el}}D}{\pi}\int_0^\infty\frac{\mathrm{d}t}{(b^2+t)\sqrt{(a^2+t)(b^2+t)t}}=\frac{F_{\mathrm{el}}D}{\pi}\frac{M(1/\zeta)}{ab^2}, \tag{31}$$

where

$$D \equiv \frac{3}{4}\left(\frac{1-\nu_1^2}{Y_1} + \frac{1-\nu_2^2}{Y_2}\right) \tag{32}$$

and $\zeta \equiv a^2/b^2$ is the ratio of the contact ellipse semi-axes. In (29)–(31) we introduce the short-hand notations[1]

$$N(\zeta) = \int_0^\infty \frac{\mathrm{d}t}{\sqrt{(1+\zeta t)(1+t)t}}\,, \tag{33}$$

$$M(\zeta) = \int_0^\infty \frac{\mathrm{d}t}{(1+t)\sqrt{(1+t)(1+\zeta t)t}}\,. \tag{34}$$

From the above relations follow the size of the contact area, a, b and the deformation ξ as functions of the elastic force F_{el} and (known) geometric coefficients B_1 and B_2.

The dependence of the force F_{el} on the deformation ξ may be obtained from scaling arguments. If we rescale $a^2 \to \alpha a^2$, $b^2 \to \alpha b^2$, $\xi \to \alpha\xi$ and $F_{\mathrm{el}} \to \alpha^{3/2} F_{\mathrm{el}}$, with α constant, eqs. (29)–(31) remain unchanged. That is, when ξ changes by the factor α, the semi-axis a and b change by the factor $\alpha^{1/2}$ and the force by the factor $\alpha^{3/2}$, *i.e.*, $a \sim \xi^{1/2}$, $b \sim \xi^{1/2}$ and

$$F_{\mathrm{el}} = \mathrm{const}\,\xi^{3/2}. \tag{35}$$

The dependence (35) holds true for all smooth convex bodies in contact. To find the constant in (35) we divide (31) by (30) and obtain the transcendental equation

$$\frac{B_2}{B_1} = \frac{\sqrt{\zeta} M(1/\zeta)}{M(\zeta)} \tag{36}$$

for the ratio of semi-axes ζ. Let ζ_0 be the root of eq. (36), then $a^2 = \zeta_0 b^2$ and we obtain from eqs. (29), (30):

$$\xi = \frac{F_{\mathrm{el}} D}{\pi}\frac{N(\zeta_0)}{b}\,, \tag{37}$$

$$B_1 = \frac{F_{\mathrm{el}} D}{\pi}\frac{M(\zeta_0)}{\zeta_0 b^3}\,, \tag{38}$$

where $N(\zeta_0)$ and $M(\zeta_0)$ are pure numbers. Equations (37) and (38) allow us to find the semi-axes b and the elastic force F_{el} as functions of the compression ξ. Hence we obtain the force, that is, we get the according constant in eq. (35) [27]

$$F_{\mathrm{el}} = \frac{\pi}{D}\left(\frac{M(\zeta_0)}{B_1 \zeta_0 N(\zeta_0)}\right)^{1/2} \xi^{3/2} = C_0\,\xi^{3/2}. \tag{39}$$

Similarly we can relate the deformation ξ and the semi-axes a of the contact ellipse [27]

$$a = \left(\frac{M(\zeta_0)}{N(\zeta_0) B_1}\right)^{1/2} \xi^{1/2}. \tag{40}$$

Note that ζ_0 is a constant determined by the collision geometry.

[1] The function $N(\zeta)$ and $M(\zeta)$ may be expressed as a combination of the Jacobian elliptic functions $E(\zeta)$ and $K(\zeta)$ [26].

For the special case of contacting spheres of the same material ($a = b$), the constants B_1 and B_2 read

$$B_1 = B_2 = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{1}{2}\frac{1}{R^{\mathrm{eff}}}\,. \tag{41}$$

In this case $\zeta_0 = 1$, $N(1) = \pi$, and $M(1) = \pi/2$, leading to the solution of (37), (38)

$$a^2 = R^{\mathrm{eff}}\,\xi, \tag{42}$$

$$F_{\mathrm{el}} = \rho\xi^{3/2}, \qquad \rho \equiv \frac{2Y}{3(1-\nu^2)}\sqrt{R^{\mathrm{eff}}}\,, \tag{43}$$

where we use the definition (32) of the constant D. This contact problem has been solved by Heinrich Hertz in 1882 [13]. It describes the force between *elastic* particles. For inelastically deforming particles it describes the repulsive force in the static case.

4 The perturbation scheme

For the most applications the viscous forces are significantly smaller than the elastic forces and the material of the bodies is rigid enough to neglect the inertial effects for impact velocities which are not very high. Let us estimate the magnitude of different terms in eq. (1). This may be easily done using the dimensionless units. For the length scale we take R, which corresponds to the characteristic size of colliding bodies, while for the time scale we use τ_c —the collision duration. Then $v_0 = R/\tau_c$ is the characteristic velocity at the impact. Taking into account that differentiation with respect to a coordinate yields for dimensionless quantities the factor $1/R$, and with respect to time – $1/\tau_c$, we obtain

$$\nabla\sigma^{\mathrm{v}} \sim \lambda_1 \nabla\sigma^{\mathrm{el}}, \qquad \lambda_1 = \tau_{\mathrm{rel}}/\tau_c, \tag{44}$$

$$\rho\ddot{u} \sim \rho\ddot{w} \sim \lambda_2 \nabla\sigma^{\mathrm{el}}, \qquad \lambda_2 = v_0^2/c^2. \tag{45}$$

Here $c^2 = Y/\rho$ and $\tau_{\mathrm{rel}} = \eta/Y$ characterize respectively the speed of sound and the microscopic relaxation time in the material and $\eta \sim \eta_1 \sim \eta_2$ [19].

Neglecting terms, of the order of λ_1 and λ_2 we get

$$\boldsymbol{\nabla}\cdot\hat{\sigma}^{\mathrm{el}}(\mathbf{u}) = 0, \qquad \boldsymbol{\nabla}\cdot\hat{\sigma}^{\mathrm{el}}(\mathbf{w}) = 0, \tag{46}$$

which yields the static displacement fields $\mathbf{u} = \mathbf{u}(\mathbf{r})$ and $\mathbf{w} = \mathbf{w}(\mathbf{r})$. This approximation corresponds to the quasi-static approximation, used in the literature [19, 21, 22, 28, 29]. Neglecting terms of the order λ_2 but keeping these of the order of λ_1, leads to the following equations:

$$\boldsymbol{\nabla}\cdot\left(\hat{\sigma}^{\mathrm{el}}(\mathbf{u}) + \hat{\sigma}^{\mathrm{v}}(\dot{\mathbf{u}})\right) = 0, \qquad \boldsymbol{\nabla}\cdot\left(\hat{\sigma}^{\mathrm{el}}(\mathbf{w}) + \hat{\sigma}^{\mathrm{v}}(\dot{\mathbf{w}})\right) = 0. \tag{47}$$

That is, to go beyond the quasi-static approximation one needs to find the solution of eqs. (47) which contains both the displacement fields $\mathbf{u}$, $\mathbf{w}$, and its time derivatives, $\dot{\mathbf{u}}$, $\dot{\mathbf{w}}$. Equations (47) need to be supplemented by the boundary conditions. These correspond to vanishing stress on the free surface and given displacement in the contact area.

In a vast majority of applications $\lambda_1 = \tau_{\rm rel}/\tau_c \ll 1$, which implies that the viscous stress is small as compared to the elastic stress. This allows to solve eq. (47) perturbatively, as a series in a small parameter. Here we follow the standard perturbation scheme, see *e.g.* [6]: To notify the order of different terms we introduce a "technical" small parameter $\lambda \sim \lambda_1$, which at the end of the computations is to be taken as one. Hence one can write

$$\hat{\sigma} = \hat{\sigma}^{(0)} + \lambda\hat{\sigma}^{(1)} + \lambda^2\hat{\sigma}^{(2)} + \dots \tag{48}$$

and, respectively,

$$\mathbf{u}(\mathbf{r}) = \mathbf{u}^{(0)}(\mathbf{r}) + \lambda\mathbf{u}^{(1)}(\mathbf{r}) + \lambda^2\mathbf{u}^{(2)}(\mathbf{r}) + \dots, \tag{49}$$

$$\mathbf{w}(\mathbf{r}) = \mathbf{w}^{(0)}(\mathbf{r}) + \lambda\mathbf{w}^{(1)}(\mathbf{r}) + \lambda^2\mathbf{w}^{(2)}(\mathbf{r}) + \dots. \tag{50}$$

Substituting the expansions (48) and (49), (50) into eq. (47) yields a set of equations for different order in λ. Zero-order equations with the according boundary conditions read

$$\begin{gathered}\boldsymbol{\nabla}\cdot\hat{\sigma}^{\rm el}\big(\mathbf{u}^{(0)}\big) = 0, \qquad \boldsymbol{\nabla}\cdot\hat{\sigma}^{\rm el}\big(\mathbf{w}^{(0)}\big) = 0,\\ B_1x^2 + B_2y^2 + u_z^{(0)}(x,y,0) + w_z^{(0)}(x,y,0) = \xi,\end{gathered} \tag{51}$$

while the first-order equations with the boundary conditions have the form

$$\begin{gathered}\boldsymbol{\nabla}\cdot\Big(\hat{\sigma}^{\rm el}\big(\mathbf{u}^{(1)}\big) + \hat{\sigma}^{\rm v}\big(\dot{\mathbf{u}}^{(0)}\big)\Big) = 0,\\ \boldsymbol{\nabla}\cdot\Big(\hat{\sigma}^{\rm el}\big(\mathbf{w}^{(1)}\big) + \hat{\sigma}^{\rm v}\big(\dot{\mathbf{w}}^{(0)}\big)\Big) = 0,\\ u_z^{(1)}(x,y,0) + w_z^{(1)}(x,y,0) = 0,\end{gathered} \tag{52}$$

and so on. Note that the zero-order equation (51) corresponds to the case of a static contact which has been considered in detail above. In particular, it yields the zero-order fields $u_z^{(0)}$ and $w_z^{(0)}$ and zero-order elastic force $F^{\rm el(0)}$, equal to that of the static case, eqs. (24), (25) and (39). This also corresponds to the quasi-static approximation widely used in the literature, *e.g.* [19,21,22,28,29]. Also note that in the proposed perturbation scheme, only zero-order problem (51) has non-zero boundary conditions, corresponding to the boundary conditions (4) of the initial problem. All other, high-order perturbation equations, have simple boundary conditions of the form, $u_z^{(k)}(x,y,0)+w_z^{(k)}(x,y,0) = 0$, $k = 1,2,\dots$. Such partition of the boundary conditions is justified due to the linearity of the problem.

Note that for the zero-order solution the condition $\sigma_{zz}^{\rm el}(\mathbf{u}^{(0)}) = \sigma_{zz}^{\rm el}(\mathbf{w}^{(0)})$ is fulfilled at the contact plane $z = 0$, as it directly follows from the construction of the solution. For the first-order solution, however, we need to additionally request the consistency condition for the first-order stress tensor:

$$\begin{gathered}\Big(\sigma_{zz}^{\rm v}\big(\dot{\mathbf{u}}^{(0)}\big) + \sigma_{zz}^{\rm el}\big(\mathbf{u}^{(1)}\big)\Big)\Big|_{z=0} =\\ \Big(\sigma_{zz}^{\rm v}\big(\dot{\mathbf{w}}^{(0)}\big) + \sigma_{zz}^{\rm el}\big(\mathbf{w}^{(1)}\big)\Big)\Big|_{z=0},\end{gathered} \tag{53}$$

which implies the equivalence of the first-order stress tensors, expressed in terms of the displacement and displacement rate of the upper and lower bodies.

5 First-order solution: Beyond quasi-static approximation

Again we will consider the upper body with $z > 0$ and introduce, for convenience, the following notations:

$$\begin{gathered}\hat{\sigma}^{\rm el}\big(\mathbf{u}^{(0)}\big) = \hat{\sigma}^{\rm el\,(0)}, \qquad \hat{\sigma}^{\rm el}\big(\mathbf{u}^{(1)}\big) = \hat{\sigma}^{\rm el\,(1)},\\ \hat{\sigma}^{\rm v}\big(\dot{\mathbf{u}}^{(0)}\big) = \hat{\sigma}^{v\,(1)}, \quad \text{etc.}\end{gathered}$$

With this notations and using eqs. (2), (3) and (11) we write

$$\begin{aligned}\sigma_{ij}^{\rm v}\big(\dot{\mathbf{u}}^{(0)}\big) = \sigma_{ij}^{\rm v(1)} &= \frac{\eta_1}{E_1}\dot{\sigma}_{ij}^{\rm el(0)}\\ &+ \left(\eta_2 - \eta_1\frac{E_2}{E_1}\right)(1+2K^{(0)})\frac{\partial \dot{f}^{(0)}}{\partial z}\delta_{ij}\end{aligned} \tag{54}$$

and accordingly the divergence of this tensor

$$\begin{aligned}\nabla_j\sigma_{ij}^{\rm v(1)} &= \left[2\eta_1K^{(0)} + (1+2K^{(0)})\left(\eta_2 + \frac{\eta_1}{3}\right)\right]\nabla_i\frac{\partial \dot{f}^{(0)}}{\partial z}\\ &= \frac{3(E_1\eta_2 - E_2\eta_1)}{(4E_1+3E_2)}\nabla_i\frac{\partial \dot{f}^{(0)}}{\partial z},\end{aligned} \tag{55}$$

where eqs. (10), (11) and eq. (13) for $K^{(0)}$ have been used. If we now apply eq. (21) for $\partial\dot{f}^{(0)}/\partial z$ and again eq. (13) for the constant $K^{(0)}$, we find the zz-component of the first-order dissipative tensor on the contact plane, $z = 0$

$$\sigma_{zz}^{\rm v(1)}(x,y,0) = \alpha\dot{\sigma}_{zz}^{\rm el(0)}(x,y,0), \tag{56}$$

$$\alpha = \frac{3\eta_2 + \eta_1}{E_1 + 3E_2}. \tag{57}$$

Similar relation may be obtained for the lower body. Using the definitions of E_1 and E_2 the coefficient α reads for each of the bodies

$$\alpha_i = \frac{(1+\nu_i)(1-2\nu_i)}{Y_i}\left(2\eta_{2(i)} + \frac{2}{3}\eta_{1(i)}\right), \tag{58}$$

where the subscript $i = 1,2$ specifies the body, $i = 1$ for the upper body and $i = 2$ for the lower one. The above relation corresponds to the according approximation of ref. [19] and coincides with the result of [21,22], where the necessary corrections have been introduced. Note, however, that the quasi-static approximation occurs to be inconsistent for the case of contact of particles of different material: Indeed, the condition (53) is possible only if the first-order elastic terms are taken into account. Obviously, this may not be achieved within the quasi-static approximation, which uses only the first-order dissipative stress $\sigma_{zz}^{\rm v(1)}$, corresponding to $\sigma_{zz}^{\rm v}(\dot{\mathbf{u}}^{(0)})$ and $\sigma_{zz}^{\rm v}(\dot{\mathbf{w}}^{(0)})$. The values of $\sigma_{zz}^{\rm v(1)}$ on the contact plane are different for the upper and lower body for different materials, since $\alpha_1 \neq \alpha_2$ (see eqs. (56)–(58)), that is, the third Newton's law for this case is violated.

Another inconsistency of the quasi-static approximation is related to vanishing dissipation for the materials with $\nu \to 1/2$, which corresponds to the substances with the elastic shear modulus much smaller than the bulk modulus[2], $E_1/E_2 = (3/2)(1-2\nu)/(1+\nu)$, as for rubber, *e.g.* [24]. In this case $(1-2\nu_i) \to 0$ and $\alpha_i \to 0$, as it follows from eq. (58). Obviously, there are no physical mechanisms in these materials that could prevent the energy dissipation.

Consider now the first-order equation (52) for the upper body

$$\nabla_j\big(\sigma_{ij}^{\mathrm{el}(1)} + \sigma_{ij}^{\mathrm{v}(1)}\big) = 0. \tag{59}$$

Due to the linearity of the problem, one can represent the first-order displacement field as a sum of two parts, $\mathbf{u}^{(1)} = \bar{\mathbf{u}}^{(1)} + \tilde{\mathbf{u}}^{(1)}$, which correspond to the two parts of the elastic tensor, $\sigma_{ij}^{\mathrm{el}(1)} = \tilde{\sigma}_{ij}^{\mathrm{el}(1)}(\tilde{\mathbf{u}}^{(1)}) + \bar{\sigma}_{ij}^{\mathrm{el}(1)}(\bar{\mathbf{u}}^{(1)})$. Here the first part of $\sigma_{ij}^{\mathrm{el}(1)}$ is the solution of the *inhomogeneous* equation with homogeneous boundary conditions

$$\nabla_j \tilde{\sigma}_{ij}^{\mathrm{el}(1)} = -\nabla_j \sigma_{ij}^{\mathrm{v}(1)}, \tag{60}$$

$$\tilde{\sigma}_{xz}^{\mathrm{el}(1)}\Big|_{z=0} = \tilde{\sigma}_{yz}^{\mathrm{el}(1)}\Big|_{z=0} = \tilde{\sigma}_{zz}^{\mathrm{el}(1)}\Big|_{z=0} = 0, \tag{61}$$

while the second part is the solution of the *homogeneous* equation

$$\nabla_j \bar{\sigma}_{ij}^{\mathrm{el}(1)} = 0, \tag{62}$$

with a given first-order displacement $u_z^{(1)}$ at the contact plane; this is to be obtained from the boundary condition (52) and consistency condition (53). The boundary problem (62) is exactly the same as the above problem (51) for the zero-order functions. Hence the same relation as eq. (24) holds true for the first-order functions, that is,

$$\bar{u}_z^{(1)}\Big|_{z=0} = -\frac{(1-\nu_1^2)}{\pi Y_1} \times \iint_S \frac{\bar{\sigma}_{zz}^{\mathrm{el}(1)}(\bar{\mathbf{u}}^{(1)}(x',y',z=0))\,\mathrm{d}x'\,\mathrm{d}y'}{|\mathbf{r}-\mathbf{r}'|}. \tag{63}$$

To solve eq. (60) we write the displacement field $\tilde{\mathbf{u}}^{(1)}$ in a form, similar to this of the zero-order solution (9)

$$\tilde{\mathbf{u}}^{(1)} = f^{(1)}\mathbf{e}_z + \nabla\varphi^{(1)}, \tag{64}$$

where $\varphi^{(1)} = K^{(1)} z f^{(1)} + \psi^{(1)}$, with $K^{(1)}$ being some constant and $f^{(1)}$ and $\psi^{(1)}$ harmonic functions. Then we can write the stress tensor $\tilde{\sigma}_{ij}^{\mathrm{el}(1)}$ as

$$\begin{aligned}\tilde{\sigma}_{ij}^{\mathrm{el}(1)} &= \big(1+2K^{(1)}\big)\Big[E_1\big(\delta_{jz}\nabla_i f^{(1)} + \delta_{iz}\nabla_j f^{(1)}\big) \\ &\quad + \left(E_2 - \frac{2}{3}E_1\right)\frac{\partial f^{(1)}}{\partial z}\delta_{ij}\Big] + 2E_1 K^{(1)} z \nabla_i\nabla_j f^{(1)} \\ &\quad + 2E_1\nabla_i\nabla_j\psi^{(1)}.\end{aligned} \tag{65}$$

If we choose $K^{(1)} = -\frac{1}{2}$ the above stress tensor takes the form

$$\tilde{\sigma}_{ij}^{\mathrm{el}(1)} = -zE_1\nabla_i\nabla_j f^{(1)} + 2E_1\nabla_i\nabla_j\psi^{(1)} \tag{66}$$

and the boundary conditions (61) read

$$\tilde{\sigma}_{xz}^{\mathrm{el}(1)}\Big|_{z=0} = \frac{\partial}{\partial x}\left(\frac{\partial\psi^{(1)}}{\partial z}\right)\Big|_{z=0} = 0, \tag{67}$$

$$\tilde{\sigma}_{yz}^{\mathrm{el}(1)}\Big|_{z=0} = \frac{\partial}{\partial y}\left(\frac{\partial\psi^{(1)}}{\partial z}\right)\Big|_{z=0} = 0. \tag{68}$$

Therefore we conclude

$$\frac{\partial\psi^{(1)}}{\partial z}\Big|_{z=0} = \mathrm{const} = 0, \tag{69}$$

where the last equation follows from the condition that $\psi^{(1)}$ vanishes at the infinity, x, $y \to \infty$, where the deformation is zero. Since $\psi^{(1)}$ is a harmonic function, we conclude that the vanishing normal derivative on a boundary, eq. (69), implies that the function vanishes everywhere, that is, $\psi^{(1)}(x,y,z) = 0$ (see *e.g.* [25]). Hence

$$\tilde{\sigma}_{ij}^{\mathrm{el}(1)} = -E_1 z\nabla_i\nabla_j f^{(1)} \tag{70}$$

and the third boundary condition, $\tilde{\sigma}_{zz}^{\mathrm{el}(1)} = 0$ at $z=0$ is automatically fulfilled. Taking into account that function $f^{(1)}$ is harmonic, we obtain

$$\nabla_j\tilde{\sigma}_{ij}^{\mathrm{el}(1)} = -E_1\nabla_i\frac{\partial f^{(1)}}{\partial z}.$$

Using the above equation together with eq. (55) we recast eq. (60) into the form

$$E_1\nabla_i\frac{\partial f^{(1)}}{\partial z} = -\frac{3(E_2\eta_1 - E_1\eta_2)}{(4E_1+3E_2)}\nabla_i\frac{\partial\dot{f}^{(0)}}{\partial z},$$

which implies the relation between functions $f^{(1)}$ and $\dot{f}^{(0)}$:

$$f^{(1)} = -\beta\dot{f}^{(0)}, \tag{71}$$

$$\beta = \frac{3(E_2\eta_1 - E_1\eta_2)}{E_1(3E_2+4E_1)}. \tag{72}$$

Using eq. (64) with $K^{(1)} = -\frac{1}{2}$ we write for $\tilde{u}_z^{(1)}$

$$\tilde{u}_z^{(1)} = \frac{1}{2}f^{(1)} - \frac{z}{2}\frac{\partial f^{(1)}}{\partial z}; \tag{73}$$

---
[2] The bulk and shear moduli respectively read, $E_2 = (1/3)Y/(1-2\nu)$ and $E_1 = (1/2)Y/(1+\nu)$ (K and μ in notations of [24]), where Y is the Young modulus and ν is the Poisson ratio. For materials with the Poisson ratio close to 1/2 ($\nu \to 1/2$) and finite bulk modulus E_2, the Young modulus Y is small. Hence for the materials with $\nu \to 1/2$ the bulk modulus E_2 is significantly larger than the shear modulus E_1.

substituting there $f^{(1)}$ from eq. (71) we arrive at

$$\tilde{u}_z^{(1)} = -\frac{1}{2}\beta\left(\dot{f}^{(0)} - z\frac{\partial \dot{f}^{(0)}}{\partial z}\right), \qquad (74)$$

where $f^{(0)}$ is given by eq. (22). Thus, the above relation presents the solution for the displacement $\tilde{u}_z^{(1)}$. Taking now into account the relation (23) between $f^{(0)}$ and $u_z^{(0)}$ at the contact plane, as well as the expression (24) for $u_z^{(0)}$ there, we find for $\tilde{u}_z^{(1)}$ at $z=0$:

$$\tilde{u}_z^{(1)} = \frac{(1-\nu_1^2)}{\pi Y_1}\iint_S \frac{\beta_1\dot{\sigma}_{zz}^{\mathrm{el}(0)}(x',y',z=0)\,\mathrm{d}x'\,\mathrm{d}y'}{|\mathbf{r}-\mathbf{r}'|}, \qquad (75)$$

where the subscript "1" indicates that the constant β_1 refers to the upper body. Similar considerations may be done for the lower body, $z<0$, yielding

$$\bar{w}_z^{(1)}\Big|_{z=0} = -\frac{(1-\nu_2^2)}{\pi Y_2} \times \iint_S \frac{\bar{\sigma}_{zz}^{\mathrm{el}(1)}(\bar{\mathbf{w}}^{(1)}(x',y',z=0))\,\mathrm{d}x'\,\mathrm{d}y'}{|\mathbf{r}-\mathbf{r}'|}, \qquad (76)$$

and

$$\tilde{w}_z^{(1)} = \frac{(1-\nu_2^2)}{\pi Y_2}\iint_S \frac{\beta_2\dot{\sigma}_{zz}^{\mathrm{el}(0)}(x',y',z=0)\,\mathrm{d}x'\,\mathrm{d}y'}{|\mathbf{r}-\mathbf{r}'|}. \qquad (77)$$

Now we apply the consistency condition (53), using eq. (56) for both bodies

$$\left(\alpha_1\dot{\sigma}_{zz}^{\mathrm{el}(0)} + \bar{\sigma}_{zz}^{\mathrm{el}(1)}(\bar{\mathbf{u}}^{(1)})\right)\Big|_{z=0} = \left(\alpha_2\dot{\sigma}_{zz}^{\mathrm{el}(0)} + \bar{\sigma}_{zz}^{\mathrm{el}(1)}(\bar{\mathbf{w}}^{(1)})\right)\Big|_{z=0}, \qquad (78)$$

where we also take into account that the following parts of the stress tensor vanish on the contact plane:

$$\tilde{\sigma}_{zz}^{\mathrm{el}(1)}(\tilde{\mathbf{u}}^{(1)})\Big|_{z=0} = \tilde{\sigma}_{zz}^{\mathrm{el}(1)}(\tilde{\mathbf{w}}^{(1)})\Big|_{z=0} = 0.$$

Equation (78) then yields

$$\bar{\sigma}_{zz}^{\mathrm{el}(1)}(\bar{\mathbf{w}}^{(1)})\Big|_{z=0} = (\alpha_1-\alpha_2)\dot{\sigma}_{zz}^{\mathrm{el}(0)}\Big|_{z=0} + \bar{\sigma}_{zz}^{\mathrm{el}(1)}(\bar{\mathbf{u}}^{(1)})\Big|_{z=0}. \qquad (79)$$

Now we use the boundary condition in eq. (52)

$$u_z^{(1)} + w_z^{(1)} = \bar{u}_z^{(1)} + \tilde{u}_z^{(1)} + \bar{w}_z^{(1)} + \tilde{w}_z^{(1)} = 0,$$

and applying eqs. (63), (75), (76) and (77) for $\bar{u}_z^{(1)}$, $\tilde{u}_z^{(1)}$, $\bar{w}_z^{(1)}$ and $\tilde{w}_z^{(1)}$ we obtain

$$\iint_S \Big[(\beta_1 D_1 + \beta_2 D_2)\dot{\sigma}_{zz}^{\mathrm{el}(0)} - D_1\bar{\sigma}_{zz}^{\mathrm{el}(1)}(\bar{\mathbf{u}}^{(1)}) - D_2\bar{\sigma}_{zz}^{\mathrm{el}(1)}(\bar{\mathbf{w}}^{(1)})\Big]\Big|_{z=0}\frac{\mathrm{d}x'\,\mathrm{d}y'}{|\mathbf{r}-\mathbf{r}'|} = 0,$$

where we introduce the short-hand notations

$$D_i = (1-\nu_i^2)/Y_i, \quad i=1,2.$$

From the above equation, together with eq. (79), the relation for the first-order elastic tensor follows:

$$\bar{\sigma}_{zz}^{\mathrm{el}(1)}(\bar{\mathbf{u}}^{(1)})\Big|_{z=0} = \left[\frac{\beta_1 D_1 + \beta_2 D_2}{D_1+D_2} - \frac{D_2(\alpha_1-\alpha_2)}{D_1+D_2}\right] \times \dot{\sigma}_{zz}^{\mathrm{el}(0)}\Big|_{z=0}. \qquad (80)$$

Finally we obtain, taking into account that the total first-order stress on the contact plane is a sum of the two parts —the elastic one, given by eq. (80), and the dissipative part $\sigma_{zz}^{\mathrm{v}(1)}$ from eq. (56), which yields,

$$\sigma_{zz}^{(1)}\Big|_{z=0} = (\sigma_{zz}^{\mathrm{v}(1)} + \sigma_{zz}^{\mathrm{el}(1)})\Big|_{z=0} = A\,\dot{\sigma}_{zz}^{\mathrm{el}(0)}\Big|_{z=0}, \qquad (81)$$

where

$$A = \frac{(\alpha_1+\beta_1)D_1 + (\alpha_2+\beta_2)D_2}{D_1+D_2}. \qquad (82)$$

Again we take into account that the component $\tilde{\sigma}_{zz}^{\mathrm{el}(1)}(\tilde{\mathbf{u}}^{(1)})$ of the stress tensor vanishes on the contact plane. The constant A may be written, using eq. (58) and (72) for α_i and β_i, as

$$A = \frac{\gamma_1 D_1 + \gamma_2 D_2}{D_1+D_2},$$

$$\gamma_i = \frac{1}{Y_i}\left(\frac{1+\nu_i}{1-\nu_i}\right)\left[\frac{4}{3}\eta_{1(i)}(1-\nu_i+\nu_i^2) + \eta_{2(i)}(1-2\nu_i)^2\right], \qquad (83)$$

which is the main result of our study. With the above eqs. (28), (39) and (40) we can write the explicit expression for the viscous pressure $P_z^{\mathrm{v}(1)} = -\sigma_{zz}^{(1)}|_{z=0}$ acting between the colliding bodies

$$P_z^{\mathrm{v}(1)}(x,y) = \frac{3AB_1}{4DM(\zeta_0)}\,\frac{\dot{a}}{\sqrt{a^2-(x^2+y^2\zeta_0)}}, \qquad (84)$$

where a depends on ξ according to eq. (40) and all other notations have been introduced in sect. 3.

6 Dissipative force

Now we can write the dissipative force acting between particles. It corresponds to the force associated with the viscous constants, that is, with the first-order stress tensor $\sigma_{zz}^{(1)}$. Integrating this stress over the contact area, we obtain, using eq. (81)

$$F_z^{\mathrm{v}(1)} = -\iint_S \sigma_{zz}^{(1)}(x,y)|_{z=0}\,\mathrm{d}x\,\mathrm{d}y = -A\frac{\partial}{\partial t}\iint_S \sigma_{zz}^{\mathrm{el}(0)}(x,y)|_{z=0}\,\mathrm{d}x\,\mathrm{d}y,$$

that is

$$F_z^{\mathrm{v}(1)} = A\dot{F}_z^{\mathrm{el}(0)}, \tag{85}$$

where $F_z^{\mathrm{el}(0)}$ is the normal force corresponding to the elastic reaction of the medium. It is equal to the Hertzian force, eq. (39); taking the time derivative of this force we finally obtain:

$$F_z^{\mathrm{v}(1)} = \frac{3}{2}AC_0\sqrt{\xi}\dot{\xi}\,. \tag{86}$$

Here the constant C_0, defined by eq. (39), is determined by the geometry of the colliding bodies and their material properties (see the discussion after eq. (39)).

Hence the total force acting between two viscoelastic bodies reads in the linear approximation with respect to the dissipative constants:

$$F_{\mathrm{tot}} = C_0\xi^{3/2} + \frac{3}{2}AC_0\sqrt{\xi}\dot{\xi}\,, \tag{87}$$

where the relation between the deformation ξ and the semi-axis a of the contact ellipse is given by eq. (40) as in the static Hertz theory. Note however, that in the Hertz theory the deformation ξ is unambiguously related to the total force acting between the bodies and, thus, one can determine the semi-axis of a contact ellipse a either by the deformation ξ or by the total force. For a visco-elastic contact, on the contrary, the size of the contact ellipse is determined by deformation ξ (or equivalently, by the elastic part of the total force), but not by the total force.

7 Conclusion

We derive a new expression for the dissipative force acting between viscoelastic bodies during an impact. Contrary to the previous theories, based on the physically plausible but non-rigorous approach, our theory exploits mathematically rigorous perturbation scheme. The small parameter in this approach is the ratio of the microscopic relaxation time and the impact duration. We compute zero and first-order terms in this perturbation expansion and find the inter-particle dissipative force. The new expression for the dissipative force noticeably differs from the previous one, obtained within the quasi-static approximation; it has a physically correct behavior for the case of colliding bodies of different materials. This has not been achievable within the quasi-static approximation. Moreover, our new theory does not suffer from the inconsistency of the previous theory with respect to materials which have the elastic bulk modulus much larger than the shear one. While the previous, quasi-static theory predicts the non-physical vanishing dissipation, the new theory implies dissipation, similar to that for "common" materials.

In the present study we neglect the inertial effects, that is, we assume that the characteristic velocity of the problem is much smaller than the speed of sound in the bodies. The general approach presented in our study may be, however, further developed to take into account the inertial effects as well as high-order terms in the perturbation series.

DSG and AVP acknowledge financial support from the Russian Science Foundation (grant no. 14-21-00090).

References

1. H.J. Herrmann, J.-P. Hovi, S. Luding (Editors), *Physics of Dry Granular Media NATO ASI Series* (Kluwer, Dordrecht, 1998).
2. H. Jaeger, S. Nagel, R. Behringer, Rev. Mod. Phys. **68**, 1259 (1996).
3. H. Hinrichsen, D.E. Wolf, *The Physics of Granular Media* (Wiley, Berlin, 2004).
4. J. Duran, *Sands, Powders and Grains* (Springer-Verlag, Berlin, 2000).
5. R. Greenberg, A. Brahic, *Planetary Rings* (The University of Arizona Press, Tucson, 1984).
6. N.V. Brilliantov, T. Pöschel, *Kinetic theory of Granular Gases* (Oxford University Press, Oxford, 2004).
7. T. Pöschel, S. Luding, *Granular Gases*, Vol. **564** of *Lecture Notes in Physics* (Springer, Berlin, 2001).
8. T. Pöschel, N.V. Brilliantov, *Granular Gas Dynamics*, Vol. **624** of *Lecture Notes in Physics* (Springer, Berlin, 2003).
9. A. Barrat, E. Trizac, M.H. Ernst, J. Phys.: Condens. Matter **17**, 2429 (2005).
10. R.D. Wildman, D.J. Parker, Phys. Rev. Lett. **88**, 064301 (2002).
11. K. Feitosa, N. Menon, Phys. Rev. Lett. **88**, 198301 (2002).
12. O. Zik, D. Levine, S. Lipson, S. Shtrikman, J. Stavans, Phys. Rev. Lett. **73**, 644 (1994).
13. H. Hertz, J. f. reine u. angewandte Math. **92**, 156 (1882).
14. T. Poeschel, T. Schwager, *Computational Granular Dynamics* (Springer, Berlin, 2005).
15. S. Luding, Nonlinearity **22**, R101 (2009).
16. T. Poeschl, Z. Phys. **46**, 142 (1928).
17. M. Montaine, M. Heckel, C. Kruelle, T. Schwager, T. Poeschel, Phys. Rev. E **84**, 041306 (2011).
18. Y.-H. Pao, J. Appl. Phys. **26**, 1083 (1955).
19. N. Brilliantov, F. Spahn, J. Hertzsch, T. Pöschel, Phys. Rev. E **53**, 5382 (1996).
20. G. Kuwabara, K. Kono, J. Appl. Phys. Part 1 **26**, 1230 (1987).
21. Q.J. Zheng, H.P. Zhu, A.B. Yu, Powder Technol. **226**, 130 (2012).
22. Q.J. Zheng, Z.Y. Zhou, A.B. Yu, Powder Technol. **248**, 25 (2013).
23. N.V. Brilliantov, A.V. Pimenova, D.S. Goldobin, EPL **109**, 14005 (2015).
24. L.D. Landau, E.M. Lifshitz, *Theory of Elasticity* (Oxford University Press, Oxford, 1965).
25. A.N. Tikhonov, A.A. Samarskii, *Equations of Mathematical Physics* (Dover Publications Inc., New York, 1963).
26. M. Abramowitz, A. Stegun, *Handbook of Mathematical Functions* (Dover Publications, 1965).
27. N.V. Brilliantov, T. Poeschel, *Collision of adhesive viscoelastic particles*, in *The Physics of Granular Media* edited by H. Hinrichsen, D. Wolf, (Wiley-VCH, Berlin, 2004).
28. N.V. Brilliantov, N. Albers, F. Spahn, T. Pöschel, Phys. Rev. E **76**, 051302 (2007).
29. E. Dintwa, M. van Zeebroeck, H. Ramon, Eur. Phys. J. B **39**, 77 (2004).